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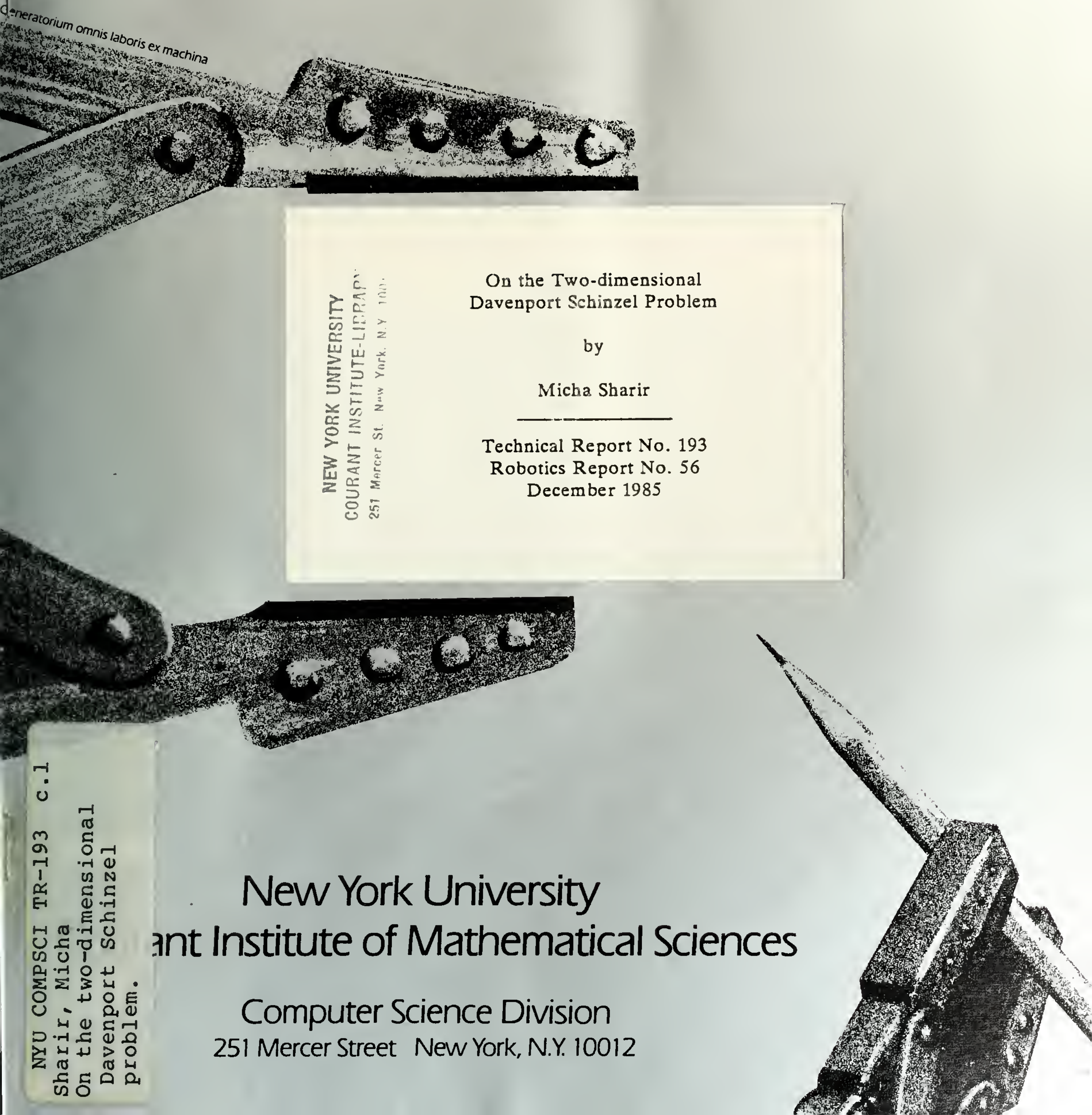

On the Two-dimensional Davenport Schinzel Problem

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Micha Sharir

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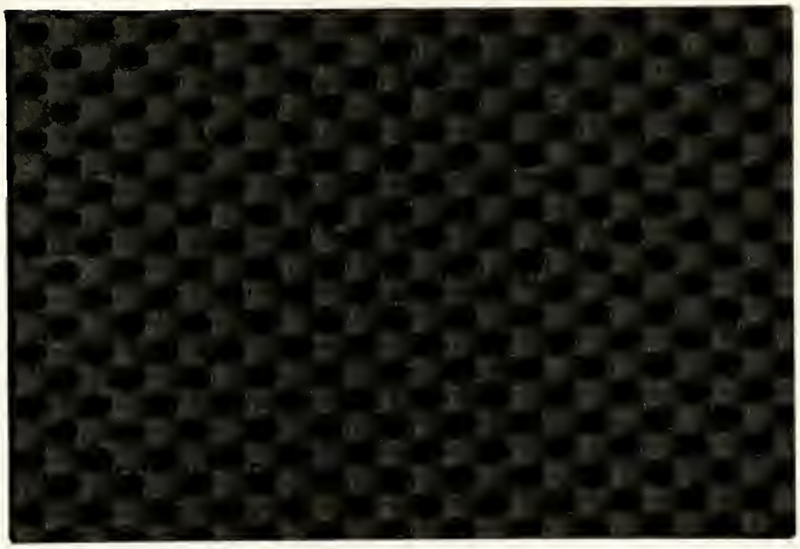

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On the Two-dimensional Davenport Schinzel Problem

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ABSTRACT

We analyze the combinatorial complexity of the minimum $M(x,y)$ of n continuous bivariate functions $f_1(x,y), \ldots, f_n(x,y)$, satisfying the conditions: (a) Each triple of functions intersect in at most s points. (b) Each pair of functions intersect in a curve having at most t singular points. (c) Each of the curves in (b) intersects every plane $x = const$ in at most two points. We show that under these assumptions the complexity of M is at most $O(n\lambda_{s+2}(n))$, where the constant of proportionality depends on s and t, and where $\lambda_r(q)$ is an almost linear function of q yielding the maximal number of connected graph portions composing the minimum of q univariate continuous functions, each pair of which intersect in at most r points.

1. Introduction

In this paper we study the problem described in the abstract, and derive the upper bound on the complexity of the minimum M stated above. This problem is a special case of the generalization to the case of bivariate functions of the following problem, initially proposed by Davenport and

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Work on this paper has been supported by Office of Naval Research Grant N00014-82-K-0381, National Science Foundation Grant No. NSF-DCR-83-20085, and by grants from the Digital Equipment Corporation, and the IBM Corporation.

Schinzel [DS]. Let $f_1(x), \ldots, f_n(x)$ be n continuous univariate functions, each pair of which intersect in at most s points. Let $\lambda_s(n)$ denote the maximal number of connected graph portions composing the minimum of such a collection of functions. It is known (cf. [DS], [At]) that $\lambda_s(n)$ is equal to the maximal length of a sequence $U = (u_1, \ldots, u_m)$ of integers which satisfies the following conditions:

(i) $1 \leq u_i \leq n$ for each i.

(ii) For each $i < m$ we have $u_i \neq u_{i+1}$.

(iii) There do not exist $s+2$ indices $1 \leq i_1 < i_2 < \cdots < i_{s+2} \leq m$ such that $u_{i_1} = u_{i_3} = u_{i_5} = \cdots = a$, $u_{i_2} = u_{i_4} = u_{i_6} = \cdots = b$, and $a \neq b$.

The problem of estimating $\lambda_s(n)$ has been studied in several papers [DS], [Da], [Sz], [At], [HS], [Sh]. It is known that $\lambda_1(n) = n$ and $\lambda_2(n) = 2n-1$. Hart and Sharir [HS] have shown that $\lambda_3(n) = \Theta(n\alpha(n))$, where $\alpha(n)$ is a functional inverse of Ackermann's function, and is thus extremely slowly growing. Szemeredi has shown [Sz] that for each $s \geq 3$, $\lambda_s(n) = O(n \log^* n)$, where the constant of proportionality depends on s, and where $\log^* n$ is the smallest $i \geq 1$ such that $\epsilon_i \geq n$, where $\epsilon_1 = 2$ and $\epsilon_{i+1} = 2^{\epsilon_i}$. Sharir [Sh] has improved this bound to $\lambda_s(n) = O(n\alpha(n)^{O(\alpha(n)^{s-3})})$.

While this "one-dimensional" problem has been solved in a satisfactory manner as described above, the two-dimensional generalization of it appears to be much harder, and is still largely unsolved. A straightforward generalization of this kind considers a collection of n continuous bivariate functions which satisfies the first two conditions given in the abstract, and aims to obtain upper and lower bounds on the maximal complexity (i.e. number of faces, edges, and vertices) of the planar map obtained by projecting the minimum of these functions onto the x,y plane. (Note that each region of this map consists of points at which M is attained by the same single function f_i, and that the edges (resp. vertices) of this map consist of points at which M is attained simultaneously by two (resp. three) functions.) A first study of this generalized problem is given by Sharir and Livne [SL], who gave a tight bound $\Theta(n^2)$ for this complexity in the case $s = 2$, and also gave a lower bound of the form $\Omega(n\lambda_3(n)) = \Omega(n^2\alpha(n))$ for this complexity in the case $s \geq 6$. The main difficulty is that this projected map does not seem to possess a simple purely combinatorial characterization, similar to that of the Davenport-Schinzel sequences given above, and this makes their analysis substantially more complex.

In this paper we obtain an upper bound of the form stated in the abstract for the special case of collections of functions which also satisfy condition (c). This extra condition is somewhat artificial, but actually arises in certain application, such as that of estimating the maximal number of free positions of a convex polygonal object, placed in a two-dimensional region amidst a collection of polygonal obstacles, in which it makes three simultaneous contacts with these obstacles. Analysis of this problem is given by Leven and Sharir [LS]. As a matter of fact, the analysis of this problem in [LS] does not exactly fit into the 2-D Davenport Schinzel problem considered here, but the techniques used in [LS] can be adapted, with some minor changes, to yield the desired upper bound for our problem, and the following analysis is thus a direct extension of that in [LS].

2. Sharp Upper Bounds for a Special Case of the 2-D Davenport-Schinzel Problem

Let $f_1(x,y), \ldots, f_n(x,y)$ be n bivariate continuous and continuously differentiable functions defined over the entire plane. We assume that the functions $f_1, \ldots, f_n$ are in *general position*, which means that (a) no pair of them intersect in a 2-dimensional patch; (b) no four of them have a nonempty intersection; and (c) each triple intersection of these functions is transversal. These assumptions about continuous differentiability and about general position of these functions are not essential, but are made to simplify the foregoing analysis. They can be dropped using standard techniques involving small perturbations.

We further assume that the functions $f_1, \ldots, f_n$ satisfy the following conditions:

I. For each $i \neq j$ and each x_0, the equation $f_i(x_0,y) = f_j(x_0,y)$ has at most two simple roots or one double root. Moreover, the plane curve $f_i(x,y) = f_j(x,y)$ has at most t singular points (including in particular points of vertical tangency, i.e. where the tangent to this curve is parallel to the y-axis).

II. For each triple i, j, k of distinct indices, the equations $f_i(x,y) = f_j(x,y) = f_k(x,y)$ have at most s roots.

Put $M(x,y) = \min_i f_i(x,y)$ for each x, y. M is the *lower envelope* of the functions $f_1, \ldots, f_n$. M induces a partition of the plane into regions, such that over each such region R, M is attained by a single function f_i. The boundary of such a region consists of points at which M is attained by at least two of the given functions. Let M^* denote the planar map obtained from this partitioning. The *complexity* of M is defined to be the complexity of M^*, i.e. the number of vertices, edges or faces of that planar map. (It follows from our assumption on general position that the number of vertices of M^*, the number of its edges and the number of its faces are all proportional to one another, so we can use any of them in measuring the complexity of M^* and thus also of M.)

Theorem 1: Under the above asumptions, the complexity of M is $O(n\lambda_{s+2}(n))$.

Proof: Let $i \neq j$ be given. We define a function $\phi_{ij}^{+}(x)$ over the entire x-axis as follows.

$$\phi_{ij}^{+}(x) = \min \{y_0 : \forall y > y_0 \;\; f_j(x,y) < f_i(x,y)\} \, .$$

(We take the value of $\phi_{ij}^{+}(x)$ to be $+\infty$ or $-\infty$ if the set of y values over which the above minimum is taken is empty or is the entire y-axis respectively.

$\phi_{ij}^{-}(x)$ is defined in the following symmetric manner.

$$\phi_{ij}^{-}(x) = \max \{y_0 : \forall y < y_0 \;\; f_j(x,y) < f_i(x,y)\} \, .$$

Define the following *i-envelopes*:

$$\psi_i^{+}(x) = \min_j \phi_{ij}^{+}(x) \qquad \text{top envelope;}$$

$$\psi_i^{-}(x) = \max_j \phi_{ij}^{-}(x) \qquad \text{bottom envelope;}$$

Obviously, for each i,x,y we have

$$y > \psi_i^+(x) \quad \Rightarrow \quad M(x,y) < f_i(x,y) \; ;$$
$$y < \psi_i^-(x) \quad \Rightarrow \quad M(x,y) < f_i(x,y) \; ;$$

In other words, the two i-envelopes define a band, monotone in the x direction, outside of which f_i cannot attain the minimum M.

Property (i) of the functions f_i implies the following

Lemma 1: Let i, j, x_0, y_0 be such that

$$f_i(x_0,y_0) = f_j(x_0,y_0) = M(x_0,y_0) \; .$$

Then y_0 must be equal to one of the values $\psi_i^+(x_0)$, $\psi_i^-(x_0)$, $\psi_j^+(x_0)$, $\psi_j^-(x_0)$.

Proof: Keep x_0 constant and let y vary. Since there are at most two values of y (one of which is y_0) at which $f_i(x_0,y) = f_j(x_0,y)$, it follows that either for every $y>y_0$ or for every $y<y_0$ one of the functions $f_i(x_0,y)$, $f_j(x_0,y)$ dominates the other. Suppose without loss of generality that for $y>y_0$ we have $f_j(x_0,y) < f_i(x_0,y)$. Then clearly $y = \phi_{ij}^+(x)$. But then we must also have $y_0 = \psi_i^+(x_0)$, for otherwise there would have to exist a k such that $y_0 > \phi_{ik}^+(x_0)$, so that

$$f_k(x_0,y_0) < f_i(x_0,y_0) = M(x_0,y_0) \; ,$$

a contradiction. □

Lemma 1 implies that points where two functions simultaneously attain the minimum must lie on some envelope. Lemma 2 below provides further characterization of points in which three of the functions attain the minimum simultaneously.

Lemma 2: Let i, j, k be a triple of distinct indices, and let x_0, y_0 be such that

$$f_i(x_0,y_0) = f_j(x_0,y_0) = f_k(x_0,y_0) = M(x_0,y_0) \; .$$

Then one of the following cases must arise, up to permutation of the indices

i,j,k, and up to replacement of top envelopes by bottom envelopes (in (i) or (iii) below):

(i) $y_0 = \psi_i^+(x_0) = \phi_{ij}^+(x_0) = \phi_{ik}^+(x_0)$;

(ii) $y_0 = \psi_i^+(x_0) = \phi_{ij}^+(x_0) = \psi_i^-(x_0) = \phi_{ik}^-(x_0)$;

(iii) $y_0 = \psi_i^+(x_0) = \phi_{ij}^+(x_0) = \psi_j^+(x_0) = \phi_{jk}^+(x_0)$ [or $= \psi_j^-(x_0) = \phi_{jk}^-(x_0)$]

$= \psi_k^+(x_0) = \phi_{ki}^+(x_0)$ [or $= \psi_k^-(x_0) = \phi_{ki}^-(x_0)$]

Proof: By applying Lemma 1 to the functions f_i, f_j, we can assume without loss of generality that

$$y_0 = \psi_i^+(x_0) = \phi_{ij}^+(x_0) .$$

Similarly, applying Lemma 1 to the functions f_i, f_k, one of the following cases arises:

(a) $y_0 = \psi_i^+(x_0) = \phi_{ik}^+(x_0)$; in this case (i) holds.

(b) $y_0 = \psi_i^-(x_0) = \phi_{ik}^-(x_0)$; in this case (ii) holds.

(c) $y_0 = \psi_k^+(x_0) = \phi_{ki}^+(x_0)$ [or $= \psi_k^-(x_0) = \phi_{ki}^-(x_0)$].

If case (c) arises, apply Lemma 1 to the functions f_j, f_k. It is easily seen that the only case which does not imply (i) or (ii) for some permutation of the indices i,j,k, and also possibly replacing top envelopes by bottom envelopes, is that in which $y_0 = \psi_j^+(x_0) = \phi_{jk}^+(x_0)$ [or $= \psi_j^-(x_0) = \phi_{jk}^-(x_0)$]. But this case plainly implies (iii). □

We now proceed to estimate the maximal number of critical points at which the minimum $M(x,y)$ is simultaneously attained by three of the functions f_i, and consider each of the cases in Lemma 2 separately.

Critical points of type (i): For each function f_i, the corresponding top envelope $\psi_i^+(x)$ is the pointwise minimum of the functions $\phi_{ij}^+(x)$, for $j \neq i$. By the assumptions on the functions $f_1, \ldots, f_n$, and by the Implicit Function Theorem, it follows that each of the functions $\phi_{ij}^+(x)$ is single-valued and continuous at each x at which $\phi_{ij}^+(x) \neq \pm\infty$, and at which the curve $f_i(x,y) = f_j(x,y)$ does not have a vertical tangent or a point of singularity at which this curve fails to be simple. Hence $\phi_{ij}^+(x)$ has at most t points of discontinuity. To simplify the foregoing analysis, we will replace each discontinuous function $\phi_{ij}^+(x)$ by at most t distinct partially defined and continuous functions, whose domains of definition are connected and pairwise disjoint. (For intervals in which $\phi_{ij}^+(x) = -\infty$, we will represent this function as a constant $-K$ for some sufficiently large K.) This still leaves us with $O(n)$ functions ϕ_{ij}^+, for each fixed i.

Moreover, for $j \neq k \neq i$, if the equation $y = \phi_{ij}^+(x) = \phi_{ik}^+(x)$ holds (where ϕ_{ij}^+, ϕ_{ik}^+ now denote restrictions of these functions to maximal connected subdomains), then we must have

$$f_i(x,y) = f_j(x,y) = f_k(x,y) ,$$

so that by condition II on these functions, each pair of the (partial and restricted) functions ϕ_{ij}^+, ϕ_{ik}^+ intersect in at most s points. It follows that the largest number of alternations of these two functions ϕ_{ij}^+, ϕ_{ik}^+ along the lower envelope ψ_i^+ is at most $s+3$ ($s+1$ alternations are possible within the intersection of their domains of definition, and two more alternations are possible within the symmetric difference of these domains). Hence the number of intersections of these functions along the top i-envelope ψ_i^+ is at most $\lambda_{s+2}(n)$ (cf. [At], [HS]). Repeating this analysis for the bottom i-envelope, and summing over all indices i, we conclude that the number of critical intersections of type (i) is $O(n\lambda_{s+2}(n))$.

Critical points of type (ii): For each index i, partition the x-axis into $O(\lambda_{s+2}(n))$ disjoint intervals, whose endpoints are all the critical values of x at which an intersection of type (i) occurs on one of the i-envelopes, or at which one of the functions $\phi_{ij}^+(x)$ or $\phi_{ij}^-(x)$ becomes singular. Let I be such an interval. Then the top i-envelope $\psi_i^+(x)$ (resp. the bottom i-envelope $\psi_i^-(x)$) is equal to a fixed function $\phi_{ij}^+(x)$ (resp. $\phi_{ik}^-(x)$) over the entire interval I. It follows that the points of intersection of $\psi_i^+(x)$ and $\psi_i^-(x)$ within I are points of intersection of a fixed pair of functions ϕ_{ij}^+, ϕ_{ik}^-, of which, arguing as above, there are at most s. Hence there are altogether at most $O(\lambda_{s+2}(n))$ intersections of type (ii) between the top and bottom i-envelopes, so that overall there are at most $O(n\lambda_{s+2}(n))$ critical intersections of type (ii).

Critical points of type (iii): Partition the x-axis into $O(n\lambda_{s+2}(n))$ disjoint intervals whose endpoints are all the critical values of x at which a critical intersection of type (i) occurs, or at which one of the functions $\phi_{ij}^+(x)$ or $\phi_{ij}^-(x)$ (for any i,j) becomes singular. Let I be one of these intervals. Then, for each index i, the top i-envelope is equal to a single function ϕ_{ij}^+ over I, and similarly for each bottom envelope.

Suppose there exists $x_0 \in I$ and three distinct indices i,j,k such that

$$\psi_i^+(x_0) = \phi_{ij}^+(x_0) = \psi_j^+(x_0) = \phi_{jk}^+(x_0) = \psi_k^+(x_0) = \phi_{ki}^+(x_0) .$$

Then, given the index i, the other two indices j,k for which the above equalities hold, must be unique and not depend on x_0. Indeed ψ_i^+ is equal to a unique function ϕ_{ij}^+ over I, thus j is uniquely determined, and, for similar reasons, so is k. Hence there are at most $O(n)$ triples i,j,k for which the above equalities can hold over I. Since each such sequence of equalities represent a point of triple intersection of f_i, f_j, f_k, it follows, repeating this argument for all possible mixtures of top and bottom envelopes, that there are at most $O(n)$ critical points of type (iii) that can be induced by the various envelopes over I. Note that even if one of these points x_0 of triple intersections lies outside I, it can still yield a type (iii) criticality, because the functions ϕ that appear in the envelopes over I which induce x_0, may still appear there also over the interval containing x_0.

Next consider how the situation changes as we pass from one such interval I to an adjacent one I' through their common endpoint x^*. Without loss of generality we can assume that at x^* only one triple intersection of type (i) takes place, and let i,j,k be the indices of the functions involved in this intersection. But then the only top or bottom envelopes that could have changed from I to I' are (some of) the i-envelopes, the j-envelopes and the k-envelopes. Thus the preceding arguments imply that only $O(1)$ new potential points at which a type (iii) critical intersection can occur, are induced by the various envelopes over I', in addition to points that have already been previously induced. Hence, proceeding through all these intervals, we obtain at most $O(n\lambda_{s+2}(n))$ critical points of type (iii).

This completes the proof of Theorem 1. □

Remarks: (1) The above analysis can easily be worked out into an algorithm for calculating M, which runs in time $O(n\lambda_{s+2}(n) \log n)$, assuming that each operation involving a specific pair or triple of the functions f_i require constant time. (Typical such operations are: find all intersections of three functions f_i, f_j, f_k; find, for a fixed x_0 the two solutions of $f_i(x_0,y) = f_j(x_0,y)$; find all the points of vertical tangency on the curve $f_i = f_j$, etc.)

(2) A typical instance of the problem which satisfies the conditions assumed in this paper is when each $f_i(x,y)$ is a polynomial in x and y of some fixed maximal degree, and is at most quadratic in y.

(3) It is interesting to note how the special case of the 2-D Davenport Schinzel problem studied in this paper barely "misses" the following useful instance of the problem: Let $f_1(x,y), \ldots, f_m(x,y)$ be a collection of m continuous bivariate functions whose graphs are polyhedral surfaces having altogether n faces. In this case the minimum $M(x,y)$ of the f_i's is also polyhedral, and its complexity, as defined above, can essentially be measured in terms of the number of (planar) faces on the graph of M. In this case we can treat each face F_i on any of the graphs of $f_1, \ldots, f_m$ as the graph of a single continuous function g_i, $i=1, \ldots, n$, where g_i pulls very steeply towards $+\infty$ outside the projection of f_i onto the x,y plane. Assuming further, without loss of generality, that each face F_i is triangular, we obtain a collection of simple polyhedral functions $g_1, \ldots, g_n$ whose minimum is also M.

The functions g_l "almost" satisfy the conditions set in this paper. Namely, each triple of them intersect in at most some fixed number of points, and the intersection curve between any two of them has at most some fixed number of singularities. However, the equation $g_l(x_0,y) = g_j(x_0,y)$, for any fixed x_0, is easily seen to have at most three solutions, rather than two as required in condition (i). Thus our technique fails to yield sharp upper bounds on the complexity of M, and it still remains an open problem whether this complexity is almost quadratic in the number of faces n.

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